JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **32**, No. 3, August 2019
http://dx.doi.org/10.14403/jcms.2019.32.3.295

ON THE HYERS-ULAM-RASSIAS STABILITY OF AN ADDITIVE-QUADRATIC-CUBIC FUNCTIONAL EQUATION

Yang-Hi Lee

Abstract. In this paper, we investigate Hyers-Ulam-Rassias stability of the functional equation

$$f(x+ky)-\frac{k^2+k}{2}f(x+y)+(k^2-1)f(x)-\frac{k^2-k}{2}f(x-y)$$
$$-f(ky)+\frac{k^2+k}{2}f(y)+\frac{k^2-k}{2}f(-y)=0.$$

1. Introduction

In this paper, let V, W be real vector spaces, X be a real normed space, Y be a real Banach space, and k be a fixed real number such that $k \neq 0, \pm 1$. For a given mapping $f : V \to W$, we use the following abbreviations:

$$\begin{aligned}
f_o(x) &:= \frac{f(x)-f(-x)}{2}, \quad f_e(x) := \frac{f(x)+f(-x)}{2},\\
Af(x,y) &:= f(x+y)-f(x)-f(y),\\
Qf(x,y) &:= f(x+y)+f(x-y)-2f(x)-2f(y),\\
Cf(x,y) &:= f(x+2y)-3f(x+y)+3f(x)-f(x-y)-6f(y),\\
D_kf(x,y) &:= f(x+ky)-\frac{k^2+k}{2}f(x+y)+(k^2-1)f(x)\\
&\quad -\frac{k^2-k}{2}f(x-y)-f(ky)+\frac{k^2+k}{2}f(y)+\frac{k^2-k}{2}f(-y)
\end{aligned}$$

Received March 13, 2019; Accepted June 21, 2019.
2010 Mathematics Subject Classification: Primary 39B82, 39B52.
Key words and phrases: stability of a functional equation, additive-quadratic-cubic functional equation, additive-quadratic-cubic mapping.

for all $x, y \in V$. Each solution of functional equation $Af(x,y) = 0$, $Qf(x,y) = 0$ and $Cf(x,y) = 0$ are called an additive mapping, a quadratic mapping and a cubic mapping, respectively. If a mapping can be expressed by the sum of an additive mapping, a quadratic mapping and a cubic mapping, then we call the mapping an additive-quadratic-cubic mapping.

A functional equation is called an additive-quadratic-cubic functional equation provided that each solution of that equation is an additive-quadratic-cubic mapping and every additive-quadratic-cubic mapping is a solution of that equation. M. E. Gordji etc. [3, 5, 6, 8] investigated the stability of the functional equation

$$f(x+ky)+f(x-ky)-k^2f(x+y)-k^2f(x-y)+2(k^2-1)f(x)=0$$

on the various spaces when k is a fixed integer. Each solution of the above equation can be expressed by the sum of a constant mapping, an additive mapping, quadratic mapping and a cubic mapping. H.-M. Kim etc.[10] and Y.-H. Lee etc. [11] investigated the stability of the above functional equation when $k = 2$, and M. E. Gordji etc. [7] investigated the stability of the above functional equation when $k = 3$.

In 1940, Ulam [14] questioned about the stability of group homomorphisms. In 1941, Hyers [9] solved this question for Cauchy functional equation, which is a partial answer to Ulam's question. In 1978, Rassias [13] made Hyers' result generalized (Refer to Găvruta's paper [2] for a more generalized result). The concept of stability used by Rassias is called 'Hyers-Ulam-Rassias stability'.

In this paper, we will show that the functional equation $D_rf(x,y) = 0$ is an additive-quadratic-cubic functional equation when r is a rational number, and also investigate Hyers-Ulam-Rassias stability of that functional equation $D_kf(x,y) = 0$ for k is a real number.

2. Main theorems

The following theorem is a particular case of Baker's theorem [1].

THEOREM 2.1. (Theorem 1 in [1]) *Suppose that V and W are vector spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ and $\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m$ are scalars such that $\alpha_j\beta_l - \alpha_l\beta_j \neq 0$ whenever $0 \leq j < l \leq m$. If $f_l : V \to W$ for $0 \leq l \leq m$ and*

$$\sum_{l=0}^{m} f_l(\alpha_l x + \beta_l y) = 0$$

for all $x, y \in V$, then each f_l is a "generalized" polynomial mapping of "degree" at most $m-1$.

Baker [1] also states that if f is a "generalized" polynomial mapping of "degree" at most $m-1$, then f is expressed as $f(x) = x_0 + \sum_{l=1}^{m-1} a_l^*(x)$ for $x \in V$, where a_l^* is a monomial mapping of degree l and $x_0 \in V$, and f has a property $f(rx) = x_0 + \sum_{l=1}^{m-1} r^l a_l^*(x)$ for $x \in V$ and $r \in \mathbb{Q}$. The monomial mapping of degree 1, 2 and 3 are also called an additive mapping, a quadric mapping and a cubic mapping, respectively.

Therefore if a mapping f satisfies the functional equation $D_k f(x,y) = 0$ for all $x, y \in V$, then f is an additive-quadratic-cubic mapping when k be a real number such that $k \neq 0, \pm 1$.

Now we will show that the functional equation $D_r f(x,y) = 0$ is an additive-quadratic-cubic functional equation when r is a rational number such that $r \neq 0, \pm 1$.

Theorem 2.2. *Let r be a rational number such that $r \neq 0, \pm 1$. A mapping f satisfies the functional equation $D_r f(x,y) = 0$ for all $x, y \in V$ if and only if f is an additive-quadratic-cubic mapping.*

Proof. If a mapping $f : V \to W$ satisfies the functional equation $D_r f(x,y) = 0$ for all $x, y \in V$, then f is an additive-quadratic-cubic mapping by Theorem 2.1.

Conversely, assume that f is an additive-quadratic-cubic mapping, i.e., there exist an additive mapping g, a quadratic mapping f' and a cubic mapping h such that $f = g + f' + h$. Notice that the equalities $g(rx) = rg(x)$, $g(x) = -g(-x)$, $f'(rx) = r^2 f'(x)$, $f'(x) = f'(-x)$, $h(rx) = r^3 h(x)$, and $h(x) = -h(-x)$ for all $x \in V$ and $r \in \mathbb{Q}$. First we know that $D_r g(x,y) = 0$ follows from the equality

$$D_r g(x,y) = Ag(x, ry) + \frac{r^2}{2} Ag(x+y, x-y) + \frac{r}{2} Ag(x+y, y-x)$$

for all $x, y \in V$. Let us first prove $D_n f'(x,y) = 0$ and $D_n h(x,y) = 0$ when n is a natural number. Using mathematical induction, the equalities $D_n f'(x,y) = 0$ and $D_n h(x,y) = 0$ are obtained from the equalities

$$\begin{aligned}
D_2 f'(x,y) =& Qf'(x+y, y) - Qf'(x,y), \quad D_2 h(x,y) = Ch(x,y), \\
D_3 f'(x,y) =& Qf'(x+2y, y) + 2D_2 f'(x,y) - Qf'(x,y), \\
D_n f'(x,y) =& Qf'(x+(n-1)y, y) + 2D_{n-1} f'(x,y) \\
& - D_{n-2} f'(x,y) - Qf'(x,y), \\
D_n h(x,y) =& D_{n-1} h(x+y, y) + \frac{n^2-n}{2} Ch(x,y)
\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Let us now prove $D_r f'(x,y) = 0$ and $D_r h(x,y) = 0$ for any rational number r with $r \neq 0, \pm 1$. Notice that if $r \in \mathbb{Q}\backslash\{0,1,-1\}$, then there exist $m, n \in \mathbb{N}$ such that $r = \frac{n}{m}$ or $r = \frac{-n}{m}$. Since the equalities $D_{\frac{n}{m}} h(x,y) = 0$, $D_{\frac{-n}{m}} h(x,y) = 0$, $D_{\frac{n}{m}} f'(x,y) = 0$ and $D_{\frac{-n}{m}} f'(x,y) = 0$ are derived from the equalities

$$\begin{aligned}
D_{\frac{n}{m}} h(x,y) =& D_n h\left(x, \frac{y}{m}\right) - \frac{n^2+mn}{2m^2} D_m h\left(x, \frac{y}{m}\right) \\
&- \frac{n^2-mn}{2m^2} D_m h\left(x, \frac{-y}{m}\right), \\
D_{\frac{-n}{m}} h(x,y) =& D_{\frac{n}{m}} h(x,-y), \\
D_{\frac{n}{m}} f'(x,y) =& D_n f'\left(x, \frac{y}{m}\right) - \frac{n^2+mn}{2m^2} D_m f'\left(x, \frac{y}{m}\right) \\
&- \frac{n^2-mn}{2m^2} D_m f'\left(x, \frac{-y}{m}\right), \\
D_{\frac{-n}{m}} f'(x,y) =& D_{\frac{n}{m}} f'(x,-y)
\end{aligned}$$

for all $x, y \in V$ and $n, m \in \mathbb{N}$, we get $D_r h(x,y) = 0$ and $D_r f'(x,y) = 0$ for all $x, y \in V$. So $D_r f(x,y) = D_r g(x,y) + D_r h(x,y) + D_r f'(x,y) = 0$ for all $x, y \in V$. $\square$

For a given mapping $f : X \to Y$, let $J_n f : X \to Y$ be the mappings defined by

$$J_n f(x) :=
\begin{cases}
4^n f_e\left(\frac{x}{2^n}\right) + \frac{4\cdot 8^n - 2^n}{3} f_o\left(\frac{x}{2^n}\right) - \frac{8^{n+1}-2^{n+3}}{3} f_o\left(\frac{x}{2^{n+1}}\right) & \text{if } p > 3, \\
4^n f_e\left(\frac{x}{2^n}\right) - \frac{2^{n-1}}{3}\left(f_o\left(\frac{x}{2^{n-1}}\right) - 8 f_o\left(\frac{x}{2^n}\right)\right) + \frac{f_o(2^{n+1}x) - 2f_o(2^n x)}{6\cdot 8^n} & \text{if } 2 < p < 3, \\
\frac{f_e(2^n x)}{4^n} - \frac{2^{n-1}}{3}\left(f_o\left(\frac{x}{2^{n-1}}\right) - 8 f_o\left(\frac{x}{2^n}\right)\right) + \frac{f_o(2^{n+1}x) - 2f_o(2^n x)}{6\cdot 8^n} & \text{if } 1 < p < 2, \\
\frac{f_e(2^n x)}{4^n} + \frac{8 f_o(2^n x) - f_o(2^{n+1}x)}{6\cdot 2^n} + \frac{f_o(2^{n+1}x) - 2f_o(2^n x)}{6\cdot 8^n} & \text{if } p < 1
\end{cases}$$

for all nonnegative integers n, and let $\Lambda f, \Delta f : X \to Y$ be the mappings defined by

$$\begin{aligned}
\Lambda f(x) :=& D_{-2} f_o(2x, -x) + 3D_{-2} f_o(x, -x), \\
\Delta f(x) :=& \frac{D_{-2} f_e(x,x)}{2}
\end{aligned} \tag{2.1}$$

when $k = -2$ and

$$\begin{aligned}\Lambda f(x) :=& \frac{1}{k^4-k^2}\Big(2D_kf_o((k-2)x,x) - 2D_kf_o((k+2)x,x)\\&+ 2D_kf_o(2x,2x) - 2D_kf_o(-2x,2x) - (k^2+k)D_kf_o(3x,x)\\&+ (k^2-k)D_kf_o(-3x,x) - 2(k^2-1)D_kf_o(-2x,x)\Big)\\&+ \frac{1}{(k^4-k^2)(k+2)}\Big(16D_kf_o(x,2x) - 16D_kf_o((k+1)x,x)\\&+ 2(k^3-2k^2-k-6)D_kf_o(2x,x) + (k^3+11k^2-6k)D_kf_o(-x,x)\\&+ 16D_kf_o(kx,x) - (k^3-23k^2-10k-16)D_kf_o(x,x)\Big),\end{aligned}$$

$$\begin{aligned}\Delta f(x) :=& \frac{1}{4k^3-4k}\bigg((k-2)\big[D_kf_e(x,2x) - D_kf_e((k+1)x,x)\\&- (k^2+k)/2D_kf_e(2x,x)\big] - (k+2)\big[D_kf_e(x,-2x)\\&- D_kf_e((k-1)x,x) - (k^2-k)/2D_kf_e(-2x,x)\big]\\&- 4D_kf_e(kx,x) + (2k^3+k^2-k-2)D_kf_e(-x,x)\\&+ (k^3-4k^2-3k+2)D_kf_e(x,x)\bigg)\end{aligned} \tag{2.2}$$

when $k \neq 0, \pm 1, -2$. By some complicated calculations we can find the equalities $\Delta f(x) = f_e(2x) - 4f_e(x)$ and $\Lambda f(x) = f_o(4x) - 10f_o(2x) + 16f_o(x)$. Then, by the definition of J_nf and the above equalities, we know that

$$J_nf(x) - J_{n+1}f(x) = \tag{2.3}$$
$$\begin{cases} 4^n\Delta f\big(\frac{x}{2^{n+1}}\big) + \frac{4\cdot 8^n}{3}\Lambda f\big(\frac{x}{2^{n+2}}\big) - \frac{2^n}{3}\Lambda f\big(\frac{x}{2^{n+2}}\big) & \text{if } p > 3,\\ 4^n\Delta f\big(\frac{x}{2^{n+1}}\big) + \frac{1}{48\cdot 8^n}\Lambda f\big(2^nx\big) - \frac{2^{n-1}}{3}\Lambda f\big(\frac{x}{2^{n+1}}\big) & \text{if } 2 < p < 3,\\ \frac{\Delta f(2^nx)}{4^{n+1}} + \frac{1}{48\cdot 8^n}\Lambda f\big(2^nx\big) - \frac{2^{n-1}}{3}\Lambda f\big(\frac{x}{2^{n+1}}\big) & \text{if } 1 < p < 2,\\ \frac{\Delta f(2^nx)}{4^{n+1}} + \frac{1}{12\cdot 2^n}\Lambda f\big(2^nx\big) - \frac{1}{48\cdot 8^n}\Lambda f\big(2^nx\big) & \text{if } p < 1\end{cases}$$

holds for all $x \in X$ and all nonnegative integers n. Therefore, together with the equality $f(x) - J_nf(x) = \sum_{i=0}^{n-1}(J_if(x) - J_{i+1}f(x))$ for all $x \in X$, we obtain the following lemma.

Lemma 2.3. *If $f : X \to Y$ is a mapping such that*

$$D_kf(x,y) = 0$$

for all $x, y \in X$, *then*

$$J_n f(x) = f(x)$$

for all $x \in X$ *and all positive integers* n.

From Lemma 2.3, we can prove the following stability theorem.

THEOREM 2.4. *Let* $p \neq 1, 2, 3$ *be a positive real number and let* k *be a fixed real number such that* $p \neq 0, \pm 1, -2$. *Suppose that* $f : X \to Y$ *is a mapping such that*

$$\|D_k f(x, y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{2.4}$$

for all $x, y \in X$. *Then there exists a unique solution mapping* F *of the functional equation* $D_k F(x, y) = 0$ *such that*

(2.5)

$$\|f(x) - F(x)\| \le \begin{cases} \left[\frac{K'}{2^p-4} + \frac{K}{3\cdot 2^p}\left(\frac{4}{2^p-8} - \frac{1}{2^p-2}\right)\right]\theta\|x\|^p & \textit{if} \quad p > 3, \\ \left[\frac{K'}{2^p-4} + \frac{K}{6}\left(\frac{1}{8-2^p} + \frac{1}{2^p-2}\right)\right]\theta\|x\|^p & \textit{if} \quad 2 < p < 3, \\ \left[\frac{K'}{4-2^p} + \frac{K}{6}\left(\frac{1}{8-2^p} + \frac{1}{2^p-2}\right)\right]\theta\|x\|^p & \textit{if} \quad 1 < p < 2, \\ \left[\frac{K'}{4-2^p} + \frac{K}{6}\left(\frac{1}{2-2^p} - \frac{1}{8-2^p}\right)\right]\theta\|x\|^p & \textit{if} \quad 0 < p < 1 \end{cases}$$

for all $x \in X$, *where*

$$\begin{aligned} K =& \frac{1}{|k^4 - k^2||k+2|}\Big(|k+2|\Big[2|k-2|^p + 4 + 2|k+2|^p \\ & + 8\cdot 2^p + |k^2+k|(3^p+1) + |k^2-k|(3^p+1) + 2|k^2-1|(2^p+1)\Big] \\ & + 16(3 + 2^p + |k|^p + |k+1|^p) + 2|k^3 - 2k^2 - k - 6|(2^p+1) \\ & + 2|k^3 + 11k^2 - 6k| + 2|k^3 - 23k^2 - 10k - 16|\Big), \\ K' =& \frac{1}{4|k^3-k|}\Big(|k-2|\Big[2 + 2^p + |k+1|^p + \frac{|k^2+k|}{2}(2^p+1)\Big] \\ & + |k+2|\Big[2^p + 2 + |k-1|^p + \frac{|k^2-k|}{2}(2^p+1)\Big] \\ & + 4(|k|^p+1) + 2|2k^3 + k^2 - k - 2| + 2|k^3 - 4k^2 - 3k + 2|\Big). \end{aligned}$$

Proof. Notice that $f(0) = 0$ is derived from $\|2(k^2-1)f(0)\| = \|D_k f(0,0)\| \le 0$. We can obtan the inequalities

$$\|\Lambda f(x)\| \le K\theta\|x\|^p, \quad \text{and} \quad \|\Delta f(x)\| \le K'\theta\|x\|^p \tag{2.6}$$

from (2.2) and (2.4). It follows from (2.3) and (2.6) that

$$\|J_nf(x)-J_{n+1}f(x)\| \\ \leq \begin{cases} \left(\frac{4^nK'}{2^{(n+1)p}}+\frac{(4\cdot 8^n-2^n)K}{3\cdot 2^{(n+2)p}}\right)\theta\|x\|^p & \text{if } p>3, \\ \left(\frac{4^nK'}{2^{(n+1)p}}+\frac{K2^{np}}{6\cdot 8^{n+1}}+\frac{2^nK}{6\cdot 2^{(n+1)p}}\right)\theta\|x\|^p & \text{if } 2<p<3, \\ \left(\frac{K'2^{np}}{4^{n+1}}+\frac{K2^{np}}{6\cdot 8^{n+1}}+\frac{2^nK}{6\cdot 2^{(n+1)p}}\right)\theta\|x\|^p & \text{if } 1<p<2, \\ \left(\frac{K'2^{np}}{4^{n+1}}+\frac{(4^{n+1}-1)2^{np}K}{6\cdot 8^{n+1}}\right)\theta\|x\|^p & \text{if } 0<p<1 \end{cases}$$

for all $x \in X$. Together with the equality

$$J_nf(x)-J_{n+m}f(x)=\sum_{i=n}^{n+m-1}(J_if(x)-J_{i+1}f(x))$$

for all $x \in X$, we get that

$$\text{(2.7)} \quad \|J_nf(x)-J_{n+m}f(x)\| \leq \\ \begin{cases} \sum_{i=n}^{n+m-1}\left(\frac{4^iK'}{2^{(i+1)p}}+\frac{(4\cdot 8^i-2^i)K}{3\cdot 2^{(i+2)p}}\right)\theta\|x\|^p & \text{if } p>3, \\ \sum_{i=n}^{n+m-1}\left(\frac{4^iK'}{2^{(i+1)p}}+\frac{K2^{ip}}{6\cdot 8^{i+1}}+\frac{2^iK}{6\cdot 2^{(i+1)p}}\right)\theta\|x\|^p & \text{if } 2<p<3, \\ \sum_{i=n}^{n+m-1}\left(\frac{K'2^{ip}}{4^{i+1}}+\frac{K2^{ip}}{6\cdot 8^{i+1}}+\frac{2^iK}{6\cdot 2^{(n+1)p}}\right)\theta\|x\|^p & \text{if } 1<p<2, \\ \sum_{i=n}^{n+m-1}\left(\frac{K'2^{ip}}{4^{i+1}}+\frac{(4^{i+1}-1)2^{ip}K}{6\cdot 8^{i+1}}\right)\theta\|x\|^p & \text{if } p<1 \end{cases}$$

for all $x \in X$ and $n, m \in \mathbb{N}\cup\{0\}$. It follows from (2.7) that the sequence $\{J_nf(x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{J_nf(x)\}$ converges for all $x \in X$. Hence we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{n\to\infty} J_nf(x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $n \to \infty$ in (2.7) we get the inequality (2.5). When $2 < p < 3$, from the definition of F, we easily get

$$
\begin{aligned}
&\|D_kF(x,y)\| \\
&= \lim_{n\to\infty} \Big\| 4^n D_k f_e\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + \frac{2^n}{6}\left(-D_k f_o\left(\frac{2x}{2^n}, \frac{2y}{2^n}\right) + 8D_k f_o\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right) \\
&\qquad + \frac{D_k f_o\left(2^{n+1}x, 2^{n+1}y\right) - 2D_k f_o\left(2^n x, 2^n y\right)}{6\cdot 8^n}\Big\| \\
&\le \lim_{n\to\infty}\left(\frac{4^n}{2^{np}} + \frac{2^n(2^p+8)}{6\cdot 2^{np}} + \frac{2^{np}(2^p+2)}{6\cdot 8^n}\right)\theta(\|x\|^p + \|y\|^p) \\
&=0
\end{aligned}
$$

for all $x, y \in X$. Also in other cases, $p < 1$ or $1 < p < 2$ or $3 < p$, we can easily see $D_kF(x,y) = 0$ in a similar way. To prove the uniqueness of F, let $F' : X \to Y$ be another solution mapping satisfying (2.5). We may replace the condition (2.5) with a simpler inequality $\|f(x) - F(x)\| \le \frac{K\theta\|x\|^p}{6}\left(\frac{1}{|8-2^p|} + \frac{1}{|2-2^p|}\right) + \frac{K'\theta\|x\|^p}{|4-2^p|}$. By Lemma 2.3, the equality $F'(x) = J_nF'(x)$ holds for all $n \in \mathbb{N}$. For the case $1 < p < 2$, we have

$$
\begin{aligned}
&\|J_nf(x) - F'(x)\| \\
&=\|J_nf(x) - J_nF'(x)\| \\
&\le\Big\|\frac{f_e(2^nx)}{4^n} - \frac{2^n}{6}\big(f_o(\frac{2x}{2^n}) - 8f_o(\frac{x}{2^n})\big) + \frac{f_o(2^{n+1}x) - 2f_o(2^nx)}{6\cdot 8^n} \\
&\quad - \frac{F_e'(2^nx)}{4^n} + \frac{2^n}{6}\big(F_o'(\frac{2x}{2^n}) - 8F_o'(\frac{x}{2^n})\big) + \frac{F_o'(2^{n+1}x) - 2F_o'(2^nx)}{6\cdot 8^n}\Big\| \\
&\le\left\|\frac{(f_e - F_e')(2^nx)}{4^n}\right\| + \frac{2^n}{6}\left\|(f_o - F_o')(\frac{2x}{2^n})\right\| + \frac{2^{n+3}}{6}\left\|(f_o - F_o')(\frac{x}{2^n})\right\| \\
&\quad + \left\|\frac{(f_o - F_o')(2^{n+1}x)}{6\cdot 8^n}\right\| + \left\|\frac{2(f_o - F_o')(2^nx)}{6\cdot 8^n}\right\| \\
&\le\left(\frac{2^{np}}{4^n} + \frac{2^{n-1}}{3\cdot 2^{(n-1)p}} + \frac{2^{n+2}}{3\cdot 2^{np}} + \frac{2^{(n+1)p}}{3\cdot 2^{3n+1}} + \frac{2^{np}}{3\cdot 2^{3n}}\right)\times \\
&\quad\left(\frac{K\theta\|x\|^p}{6}\big(\frac{1}{|8-2^p|} + \frac{1}{|2-2^p|}\big) + \frac{K'\theta\|x\|^p}{|4-2^p|}\right) \\
&=\frac{K\theta\|x\|^p}{6}\big(\frac{1}{|8-2^p|} + \frac{1}{|2-2^p|}\big) + \frac{K'\theta\|x\|^p}{|4-2^p|}
\end{aligned}
$$

for all $x \in X$ and all positive integer n. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F'(x) = \lim_{n\to\infty} J_nf(x)$ for all $x \in X$. Also in a different case, $0 < p < 1$ or $2 < p < 3$ or $3 < p$, we can easily see that $F'(x) = \lim_{n\to\infty} J_n f(x)$ in a similar way. This means that $F(x) = F'(x)$ for all $x \in X$. □

Theorem 2.5. *Let $p \leq 0$ be a real number. Suppose that $f : X \to Y$ is a mapping satisfying the inequality (2.4) for all $x, y \in X\backslash\{0\}$ and $f(0) = 0$. If $p = 0$, then there exists a unique solution mapping F of the functional equation $D_k F(x, y) = 0$ such that*

$$\|f(x) - F(x)\| \leq \Big[\frac{K'}{3} + \frac{K}{7}\Big]\theta \tag{2.8}$$

for all $x \in X\backslash\{0\}$. If $p < 0$, then f is s solution of the functional equation $D_k f(x, y) = 0$.

Proof. Notice that $D_k f(0, 0) = 0$ is derived from $D_k f(0, 0) = 2(k^2 - 1)f(0)$ and $f(0) = 0$. Notice that (2.6) holds for all $x \in X\backslash\{0\}$. It follows from (2.3) and (2.6) that

$$\|J_n f(x) - J_{n+1} f(x)\| \leq \Big(\frac{K'2^{np}}{4^{n+1}} + \frac{(4^{n+1} - 1)2^{np}K}{6 \cdot 8^{n+1}}\Big)\theta\|x\|^p$$

for all $x \in X\backslash\{0\}$. Together with the equality $J_n f(x) - J_{n+m} f(x) = \sum_{i=n}^{n+m-1}(J_i f(x) - J_{i+1} f(x))$ for all $x \in X$, we get

$$\|J_n f(x) - J_{n+m} f(x)\| \leq \sum_{i=n}^{n+m-1} \Big(\frac{K'2^{ip}}{4^{i+1}} + \frac{(4^{i+1} - 1)2^{ip}K}{6 \cdot 8^{i+1}}\Big)\theta\|x\|^p \tag{2.9}$$

for all $x \in X\backslash\{0\}$ and $n, m \in \mathbb{N} \cup \{0\}$. It follows from (2.9) that the sequence $\{J_n f(x)\}$ is a Cauchy sequence for all $x \in X\backslash\{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_n f(x)\}$ converges for all $x \in X\backslash\{0\}$. Hence we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{n\to\infty} J_n f(x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $n \to \infty$ in (2.9) we get the inequality

$$\|f(x) - F(x)\| \leq \Big[\frac{K'}{4 - 2^p} + \frac{K}{(2 - 2^p)(8 - 2^p)}\Big]\theta\|x\|^p \tag{2.10}$$

for all $x \in X\backslash\{0\}$. In particular, the inequality (2.8) holds for all $x \in X\backslash\{0\}$. From the definition of F, we easily get

$$
\begin{aligned}
&\|D_kF(x,y)\| \\
&= \lim_{n\to\infty} \left\| \frac{D_kf_e\left(2^nx,2^ny\right)}{4^n} + \frac{8D_kf_o\left(2^nx,2^ny\right) - D_kf_o\left(2^{n+1}x,2^{n+1}y\right)}{6\cdot 2^n} \right. \\
&\quad \left. + \frac{D_kf_o\left(2^{n+1}x,2^{n+1}y\right) - 2D_kf_o\left(2^nx,2^ny\right)}{6\cdot 8^n} \right\| \\
&\le \lim_{n\to\infty} \left(\frac{2^{np}}{4^n} + \frac{2^{np}(2^p+8)}{6\cdot 2^n} + \frac{2^{np}(2^p+2)}{6\cdot 8^n} \right) \theta(\|x\|^p + \|y\|^p) \\
&=0
\end{aligned}
$$

for all $x, y \in X\backslash\{0\}$. Also $D_kF(x,y) = 0$ for all $x, y \in X$ is derived from $f(0) = 0$ and $D_kF(x,y) = 0$ for all $x, y \in X\backslash\{0\}$. To prove the uniqueness of F, let $F' : X \to Y$ be another solution mapping satisfying (2.10) for all $x \in X\backslash\{0\}$. By Lemma 2.3, the equality $F'(x) = J_nF'(x)$ holds for all $n \in \mathbb{N}$. For the case $p = 0$, we have

$$
\begin{aligned}
&\|J_nf(x) - F'(x)\| \\
&=\|J_nf(x) - J_nF'(x)\| \\
&\left\| \frac{f_e(2^nx)}{4^n} - \frac{F'_e(2^nx)}{4^n} + \frac{8f_o(2^nx) - f_o(2^{n+1}x)}{6\cdot 2^n} - \frac{8F'_o(2^nx) - F'_o(2^{n+1}x)}{6\cdot 2^n} \right. \\
&\quad \left. + \frac{f_o(2^{n+1}x) - 2f_o(2^nx)}{6\cdot 8^n} - \frac{F'_o(2^{n+1}x) - 2F'_o(2^nx)}{6\cdot 8^n} \right\| \\
&\le \left(\frac{2^{np}}{4^n} + \frac{2^{np}(8+2^p)}{6\cdot 2^n} + \frac{2^{np}(2+2^p)}{6\cdot 8^n} \right) \left(\frac{K'}{3} + \frac{K}{7} \right) \theta
\end{aligned}
$$

for all $x \in X\backslash\{0\}$ and all positive integer n. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F'(x) = \lim_{n\to\infty} J_nf(x)$ for all $x \in X$. This means that $F(x) = F'(x)$ for all $x \in X$.

To show the equality $F = f$ if $p < 0$, assume that $F : X \to Y$ is a solution mapping of $D_kF(x,y) = 0$ satisfying the condition (2.10) for all

$x \in X\backslash\{0\}$. From the equality

$$\begin{aligned}
&D_k f((n+1)x, nx) \\
&\quad = D_k f((n+1)x, nx) - D_k F((n+1)x, nx) \\
&\quad = (f-F)((n+1)x + knx) - \frac{k^2+k}{2}(f-F)((n+1)x + nx) \\
&\qquad + (k^2-1)(f-F)((n+1)x) - \frac{k^2-k}{2}(f-F)(x) \\
&\qquad - (f-F)(knx) + \frac{k^2+k}{2}(f-F)(nx) + \frac{k^2-k}{2}(f-F)(-nx)
\end{aligned}$$

for all $x \in X\backslash\{0\}$ and $n \in \mathbb{N}$, we have the inequality

$$\begin{aligned}
&\frac{|k^2-k|}{2}\|(f-F)(x)\| \\
&\quad = \Bigg\|(f-F)((kn+n+1)x) - \frac{k^2+k}{2}(f-F)((2n+1)x) \\
&\qquad + (k^2-1)(f-F)((n+1)x) + \frac{k^2+k}{2}(f-F)(nx) \\
&\qquad - (f-F)(knx) + \frac{k^2-k}{2}(f-F)(-nx) - D_k f((n+1)x, nx)\Bigg\| \\
&\quad \le \Bigg[\left(\frac{K'}{4-2^p} + \frac{K}{(2-2^p)(8-2^p)}\right)\big(|kn+n+1|^p + |kn|^p + k^2 n^p \\
&\qquad + \frac{|k^2+k|}{2}(2n+1)^p + |k^2-1|(n+1)^p\big) + n^p + (n+1)^p\Bigg]\theta\|x\|^p
\end{aligned}$$

for all $x \in X\backslash\{0\}$ and $n \in \mathbb{N}$. Since $n^p + (n+1)^p$ and $|kn+n+1|^p + |kn|^p + k^2n^p + \frac{|k^2+k|}{2}(2n+1)^p + |k^2-1|(n+1)^p) + (n^p + (n+1)^p)$ tend to 0 as $n \to \infty$, we get $f(x) = F(x)$ for all $x \in X$ from $f(0) = F(0)$. $\square$

Using the equalities (2.1), we can show the following theorems in the same way that we have proved Theorem 2.4 and Theorem 2.5, so the proof is omitted and described only.

Theorem 2.6. *Let $p \neq 1, 2, 3$ be a positive real number. Suppose that $f : X \to Y$ is a mapping such that*

$$\|D_{-2} f(x,y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{2.11}$$

for all $x, y \in X$. *Then there exists a unique solution mapping* F *of the functional equation* $D_{-2}F(x, y) = 0$ *such that*

$$\|f(x) - F(x)\| \leq \left(\frac{1}{|2^p - 4|} + \frac{2^p + 7}{|2^p - 8||2^p - 2|}\right)\theta\|x\|^p$$

for all $x \in X$.

THEOREM 2.7. *Let* $p \leq 0$ *be a real number. Suppose that* $f : X \to Y$ *is a mapping satisfying the inequality* (2.11) *for all* $x, y \in X\backslash\{0\}$ *and* $f(0) = 0$. *If* $p = 0$, *then there exists a unique solution mapping* F *of the functional equation* $D_{-2}F(x, y) = 0$ *such that*

$$\|f(x) - F(x)\| \leq \frac{31}{21}\theta$$

for all $x \in X\backslash\{0\}$. *If* $p < 0$, *then* f *is a solution of the functional equation* $D_{-2}f(x, y) = 0$.

References

[1] J. Baker, *A general functional equation and its stability*, Proc. Natl. Acad. Sci., **133** (2005), no. 6, 1657–1664.

[2] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (1994), 431–436.

[3] M. E. Gordji, M. Ghanifard, H. Khodaei, and C. Park, *Fixed points and the random stability of a mixed type cubic, quadratic and additive functional equation*, J. Comput. Anal. Appl., **15** (2013), 612–621.

[4] M. E. Gordji, S. K. Gharetapeh, J. M. Rassias, and S. Zolfaghari, *Solution and stability of a mixed type additive, quadratic, and cubic Functional Equation*, Adv. Difference Equ. **2009** Article ID 826130, 17 pages.

[5] M. E. Gordji, M. Kamyar, H. Khodaei, D. Y. Shin and C. Park, *Fuzzy stability of generalized mixed type cubic, quadratic, and additive functional equation*, J. Inequal. Appl., **2011**, 2011:95, 22 pages.

[6] M. E. Gordji and H. Khodaie, *Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces*, Nonlinear Anal., **71** (2009), 5629–5643.

[7] M. E. Gordji and M. B. Savadkouhi, *Stability of a mixed type additive, quadratic, and cubic functional equation in random normed spaces*, Filomat, **25** (2011), 43–54.

[8] M. E. Gordji, M. B. Savadkouhi, and Th. M. Rassias, *Stability of generalized mixed type additive-quadratic-cubic functional equation in non-Archimedean spaces*, arXiv preprint arXiv:0909.5692, (2009).

[9] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A., **27** (1941), 222–224.

[10] K.-W. Jun and H.-M. Kim, *On the Hyers-Ulam-Rassias stability of a general cubic functional equation*, Math. Inequal. Appl., **6** (2003), 289–302.

[11] Y.-H. Lee, *On the generalized Hyers-Ulam stability of the generalized polynomial function of degree 3*, Tamsui Oxf. J. Math. Sci., **24** (2008), no. 4, 429–444.
[12] C. K. Park, *A fixed point approach to the fuzzy stability of an additive-quadratic-cubic functional equation*, Fixed Point Theory and Applications, **2009** (2009), Article ID 918785, 24 pages.
[13] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300.
[14] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.

Department of Mathematics Education
Gongju National University of Education
Gongju 32553, Korea
E-mail: lyhmzi@pro.gjue.ac.kr